B.sc Part I (Hons)

Paper II (Leibnitz theorem)

**Leibnitz Theorem:-** If $u$ and $v$ are two functions of $x$ which possess derivatives of $n$th order then

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots +$$

$${}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n.$$

**Proof:-** By method of induction

Let $y = uv$

Where $u, v$ are functions of $x$.

By directly differentiating successively, we get

$$y_1 = u_1 v + u v_1$$

$$y_2 = (u_2 v + u_1 v_1) + (u v_2 + u_1 v_1)$$

$$= u_2 v + 2 u_1 v_1 + u v_2 = u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2.$$

$$y_3 = (u_3 v + u_2 v_1) + 2(u_1 v_2 + u_2 v_1) + (u v_3 + u_1 v_2)$$

$$= u_3 v + 3 u_2 v_1 + 3 u_1 v_2 + u v_3$$

$$= u_3 v + {}^3C_1 u_2 v_1 + {}^3C_2 u_2 v_2 + {}^3C_3 u v_3.$$

Thus we see that this theorem is true for $n = 1, 2, 3$. According to the law of induction, we assume that this theorem is true for $n = m$ and we shall prove that this will also be true for $n = m+1$ and since this is true for particular values of $n = 1, 2, 3$, therefore it will be true for every value of $n$.

Now we assume that this theorem holds for $n = m$ i.e. we shall get the same formal expression for $y_m$ which will be obtained by putting $n = m$ in the statement of the theorem that is,

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \cdots$$

$$\cdots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \cdots +$$

$${}^nC_{m-1} u_1 v_{m-1} + {}^mC_m u v_m \quad \text{------} (1)$$

Differentiating once, we get

$$y_{m+1} = (u_{m+1}v + u_m v_1) + {}^mc_1(u_m v_1 + u_{m-1}v_2)$$

$$+ {}^mc_2(u_{m-1}v_2 + u_{m-2}v_3) + \cdots + {}^mc_{r-1}(u_{m-r+2}$$

$$u_{m-r+1}v_r) + {}^mc_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) +$$

$$\cdots + {}^mc_{m-1}(u_2 v_{m-1} + u_1 v_m) + {}^mc_m(u_1 v_m + u v_{m+1})$$

$$= u_{m+1}v + u_m v_1({}^mc_0 + {}^mc_1) + u_{m-1}v_2({}^mc_1 + {}^mc_2) + \cdots$$

$$\cdots + u_{m-r+1}v_r({}^mc_{r-1} + {}^mc_r) + \cdots$$

$$+ u_1 v_m({}^mc_{m-1} + {}^mc_m) + {}^mc_m u v_{m+1}.$$

But we know that:

$${}^mc_{r-1} + {}^mc_r = {}^{m+1}c_r.$$

$\therefore$ putting $r = 1, 2, 3, \cdots$

$${}^mc_0 + {}^mc_1 = {}^{m+1}c_1,\quad {}^mc_1 + {}^mc_2 = {}^{m+1}c_2 \text{ etc.}$$

Hence
$$y_{m+1} = u_{m+1}v + {}^{m+1}c_1 u_m v_1 + {}^{m+1}c_2 u_{m-1}v_2$$

$$+ \cdots + {}^{m+1}c_r u_{m-r+1}v_r + {}^{m+1}c_{m+1} u v_{m+1}.$$

Thus we see that if we assume that theorem to be true for a particular value of $n=m$, then this theorem is also true for the next higher integer $n=m+1$.

But we have shown before that this theorem is true for $n=2,3$, therefore it is true for $n=4$ and since this is true for $n=4$, hence this is true for $n=5$.

Hence this theorem is true for every integral value of $n$. Thus the theorem is proved: Anjani kumar singh